Arithmetic structure of Mumford's fake projective plane [1]

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We prove that Mumford's fake projective plane is a connected Shimura variety associated to a certain unitary group. We also describe explicitly the unitary group and the necessary data for defining this Shimura variety, and give a field of definition.

1. Introduction

In this paper we construct explicitly a Shimura surface of PEL-type whose connected components are, as complex surfaces, isomorphic to Mumford's fake projective plane. A fake projective plane is a non-singular projective surface of general type having the same topological Betti numbers as $\mathbb{CP}^2$. The first example of fake projective planes was discovered by Mumford in [Mu79], where he constructed it by using the theory of p-adic uniformization. Later in part of [IK98] Masanori Ishida and the author observed other possible fake projective planes according to Mumford's idea, and proved there exist (at least) two more different fake projective planes. As far as the author knows, these three are the only known fake projective planes.

A fake projective plane has complex unit-ball as the topological universal covering, and in this sense it is among the most interesting classes of algebraic surfaces. However, very little is known about its algebraic or analytic structure (recently Barlow [Ba99] investigated zero-cycles on Mumford's fake projective plane in connection with Bloch conjecture). For instance, topological or even some algebro-geometric properties of them (e.g. fundamental groups, field of definition, etc.) are quite obscure; even the arithmeticity of the fundamental groups seems unknown. The main reason for this is that the known fake projective planes are, a priori, constructed as surfaces over the field of 2-adic numbers $\mathbb{Q}_2$, not over $\mathbb{C}$, and hence are not quite concrete as complex surfaces. (A fake projective plane over an arbitrary field may be defined to be a smooth surface enjoying: $\mathrm{c}_1^2 = 3\mathrm{c}_2 = 9$, $\mathrm{p_g} = \mathrm{q} = 0$ and the canonical class is ample; if one considers this condition in the complex analytic category, this condition is equivalent to that the surface in question is a fake projective plane in the origial sense.)

The theory of p-adic uniformization, on the other hand, involves surprizingly rich algebro-geometric and arithmetic aspects concerning with Shimura varieties. This was first observed by Čerednik [Če76] and Drinfeld [Dr76] for Shimura curves. The generalizations to higher dimensions have been developed over the past two decades; recently, Rapoport-Zink [RZ96], Boutot-Zink [BZ95] and Varshavsky [Va98a][Va98b] gave powerfully extended generalities. These theories give a nice symmetric picture involving p-adic and complex uniformizations of certain Shimura varieties of PEL-type; viz. such a Shimura variety has both p-adic (for very special p) and complex analytic uniformizations, and they can be viewed symmetrically by means of

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"changing-invariants" trick. Shimura varieties of this kind are therefore much expected to possess rich structures involving the interplay between p-adic and complex analysis. However, as far as the author knows, very few such examples are known in dimension two or high, in contract with numerous explicit examples of Shimura curves of this type (see, e.g. [Ku79][Ku94][An98]).

Sketch of Results. In the next section we will construct explicitly a Shimura variety $\mathcal{S}h_C$ associated to a certain central division algebra over $K = \mathbb{Q}(\sqrt{-7})$. The Shimura variety $\mathcal{S}h_C$ has the canonical model Sh_C over K. We will apply the theory by Rapoport-Zink [RZ96], Boutot-Zink [BZ95] and Varshavsky [Va98a][Va98b] to this Shimura variety to show the following results (Theorem 4.3 and Corollary 4.4):

Theorem. *The canonical model* Sh_C *of the Shimura variety* $\mathcal{S}h_C$ *has exactly three geometrically connected components defined over* $\mathbb{Q}(\zeta_7)$ *which are permuted by the action of* $\mathrm{Gal}(\mathbb{Q}(\zeta_7)/\mathbb{Q}(\sqrt{-7}))$*. Moreover, for any connected component* S *of* Sh_C *there exists an isomorphism*

$$X_{\mathrm{Mum}} \times_{\mathrm{Spec}\,\mathbb{Q}_2} \mathrm{Spec}\,\mathbb{Q}_2(\zeta_7) \xrightarrow{\sim} S \times_{\mathrm{Spec}\,\mathbb{Q}(\zeta_7)} \mathrm{Spec}\,\mathbb{Q}_2(\zeta_7)$$

over $\mathbb{Q}_2(\zeta_7)$*, where* X_{Mum} *is the Mumford's fake projective plane.*

Corollary. *Each connected component of the complex surface* $\mathcal{S}h_C$ *is a fake projective plane, which has a field of definition* $\mathbb{Q}(\zeta_7)$*. Moreover, it is an arithmetic quotient of complex unit-ball.*

The structure of this paper is as follows: In the next section we construct the Shimura variety $\mathcal{S}h$ by presenting the necessary data explicitly. In §2 we review briefly Mumford's construction of the fake projective plane. In the final section §4 we investigate the relation between the fake projective plane and the Shimura variety $\mathcal{S}h$, and prove the main theorem.

Notation and conventions. The notation $A \otimes B$ (absence of the base ring) only occurs when A and B are $\mathbb{Q}$-algebra, and the tensor is taken over $\mathbb{Q}$. For a field extension F/E, we denote by $\mathrm{Res}_{F/E}$ the Weil restriction. For a ring A we denote by A^{op} the opposite ring. By an involution of a ring A we always mean a homomorphism $*\colon A \to A^{\mathrm{op}}$ such that $* \circ * = \mathrm{id}_A$.

2. Construction of Shimura variety

2.1. In this section we construct a Shimura variety which will later be related to Mumford's fake projective plane. Our Shimura variety $\mathcal{S}h$ is of PEL-type related to a certain unitary group. In order to define it we introduce

- a central division algebra D over K,
- a positive involution $*$ of D of second kind,
- and a non-degenerate anti-hermitian form ψ.

These data yield the Shimura data (cf. [De71]) of our Shimura variety.

Let $\zeta = \zeta_7$ be a primitive 7th-root of unity, and set $L = \mathbb{Q}(\zeta)$; L is a cyclic extension of $\mathbb{Q}$ of degree 6 having the intermediate quadratic extension $K = \mathbb{Q}(\lambda)$ ($\cong \mathbb{Q}(\sqrt{-7})$), where $\lambda = \zeta + \zeta^2 + \zeta^4$. The Galois group of L/K is generated by the

Frobenius map $\sigma\colon \zeta \mapsto \zeta^2$, and that of $K/\mathbb{Q}$ is generated by the complex conjugation $z \mapsto \overline{z}$. Note that the prime 2 decomposes on K such that $2 = \lambda\overline{\lambda}$ gives the prime factorization. The prime 7 ramifies on K. We fix an infinite place (CM-type) $\varepsilon\colon K \hookrightarrow \mathbb{C}$ by $\lambda \mapsto \frac{-1+\sqrt{-7}}{2}$. Note that the class number of K is 1. We set

$$\mu = \lambda/\overline{\lambda}.$$

Now the central division algebra D over K (of dimension 9) is defined by

$$D = \bigoplus_{i=0}^{2} L\Pi^i; \quad \Pi^3 = \mu,\ \Pi z = z^{\sigma}\Pi \ \ (z \in L).$$

2.1.1. Lemma. *The L-algebra $D \otimes_K L$ is isomorphic to the matrix algebra $\mathrm{M}_3(L)$. In particular, we have $D_\varepsilon \cong \mathrm{M}_3(\mathbb{C})$.*

This is well-known; for the later purpose we give an isomorphism $\Phi\colon D \otimes_K L \xrightarrow{\sim} \mathrm{M}_3(L)$ explicitly as follows: Set $V = D$, which we consider as a left D-module. Then we have

$$V \otimes_K L \xrightarrow{\sim} \bigoplus_{i=0}^{2} V \otimes_{L,\sigma^i} L.$$

Looking at the action of D on the factor $V \otimes_{L,\sigma^0} L$ subject to the base $\{1\otimes 1, \Pi \otimes 1, \Pi^2 \otimes 1\}$ we obtain a matrix presentation of elements in D determined by

$$z \mapsto \begin{bmatrix} z & & \\ & z^{\sigma^2} & \\ & & z^{\sigma} \end{bmatrix} \quad \text{and} \quad \Pi \mapsto \begin{bmatrix} & & \mu \\ 1 & & \\ & 1 & \end{bmatrix}, \tag{2.1.2}$$

which induces the desired isomorphism Φ. By taking an embedding $\theta\colon L \hookrightarrow \mathbb{C}$ by $\zeta \mapsto \exp\frac{2i\pi}{7}$ we can extend Φ to an isomorphism $D_\varepsilon \xrightarrow{\sim} \mathrm{M}_3(\mathbb{C})$.

2.1.3. Lemma. *We have* $\mathrm{inv}_\lambda D = 1/3$ *and* $\mathrm{inv}_{\overline{\lambda}} D = -1/3$, *while for any finite place ℓ which does not divide* 2 *we have* $\mathrm{inv}_\ell D = 0$.

This is clear by the construction. We define the order $\mathcal{O}_D$ of D by

$$\mathcal{O}_D = \mathcal{O}_L \oplus \mathcal{O}_L\overline{\lambda}\Pi \oplus \mathcal{O}_L\overline{\lambda}\Pi^2. \tag{2.1.4}$$

It is easy to verify that $\mathcal{O}_{D_\lambda} = \mathcal{O}_D \otimes_{\mathcal{O}_K} \mathcal{O}_{K_\lambda}$ and $\mathcal{O}_{D_{\overline{\lambda}}} = \mathcal{O}_D \otimes_{\mathcal{O}_K} \mathcal{O}_{K_{\overline{\lambda}}}$ are maxial orders of $D_\lambda = D \otimes_K K_\lambda$ and $D_{\overline{\lambda}} = D \otimes_K K_{\overline{\lambda}}$, respectively.

2.2. Next we define an involution (positive of second kind) $*$ on D by

$$\Pi^* = \overline{\mu}\Pi^2 \quad \text{and} \quad z^* = \overline{z} \ \ (z \in L).$$

It is easy to check that the involution $*$ is mapped by the isomorphism Φ as above to the standard involution $(a_{ij})^* = (\overline{a}_{ji})$. Set

$$b = (\lambda - \overline{\lambda}) - \overline{\lambda}\Pi + \overline{\lambda}\Pi^2. \tag{2.2.1}$$

The element b is anti-symmetric (i.e. $b^* = -b$); we have

$$\Phi(b) = \begin{bmatrix} \lambda - \overline{\lambda} & \lambda & -\lambda \\ -\overline{\lambda} & \lambda - \overline{\lambda} & \lambda \\ \overline{\lambda} & -\overline{\lambda} & \lambda - \overline{\lambda} \end{bmatrix}. \tag{2.2.2}$$

The element b induces an involution, positive of second kind, $\bigstar\colon D \to D^{\mathrm{op}}$ by $\alpha^\bigstar = b\alpha^* b^{-1}$ and a $\mathbb{Q}$-bilinear form $\psi\colon V \times V \to \mathbb{Q}$ on V by

$$\psi(x, y) = \mathrm{tr}_{D/\mathbb{Q}}(ybx^*)$$

for $x, y \in V$.

2.3. Lemma.

1) *The bilinear form ψ is non-degenerate and anti-symmetric. Moreover we have $\psi(\alpha x, y) = \psi(x, \alpha^* y)$ for $\alpha \in D$ and $x, y \in V$.*

2) *There exists $P \in \mathrm{M}_3(\mathbb{C})$ such that $P^*\Phi(b)P = \mathrm{diag}(-\sqrt{-1}, -\sqrt{-1}, \sqrt{-1})$.*

Proof. 1) Non-degeneracy is clear since b is invertible. The anti-symmetricity follows from the equality $b = -b^*$. The other assertion is clear. The characteristic polynomial of $\Phi(b)$ is $t^3 - 3\sqrt{-7}t^2 - 15t - \sqrt{-7}$. By this 2) follows immediately. □

2.4. Definition. We define an algebraic group G over $\mathbb{Q}$ by

$$\begin{aligned} G(R) &= \{\gamma \in (D \otimes R)^\times \mid \psi(\gamma x, \gamma y) = c(\gamma)\psi(x, y),\ c(\gamma) \in R^\times,\ x, y \in V \otimes R\} \\ &= \{\gamma \in (D \otimes R)^\times \mid \gamma\gamma^\bigstar \in R\} \end{aligned}$$

for any (commutative) $\mathbb{Q}$-algebra R.

2.4.1. Lemma. *The isomorphism $\Phi\colon D \otimes \mathbb{R} \xrightarrow{\sim} \mathrm{M}_3(\mathbb{C})$ defined by (2.1.2) induces an isomorphism $\Phi\colon G_{\mathbb{R}} \xrightarrow{\sim} \mathrm{GU}(2, 1)$.*

This is clear by 2.3.2.

2.5. Consider the homomorphism of algebraic groups over $\mathbb{R}$

$$h\colon \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_{\mathrm{m}} \longrightarrow G_{\mathbb{R}} \tag{2.5.1}$$

such that $\Phi \circ h(\sqrt{-1}) = \mathrm{diag}(\sqrt{-1}, \sqrt{-1}, -\sqrt{-1})$ and $\Phi \circ h(r)$ is the scalar matrix r for $r \in \mathbb{R}^\times$. By [Ko92, 4.1] the pair $(G_{\mathbb{R}}, h)$ satisfies the conditions (1.5.1), (1.5.2) and (1.5.3) in [De71], and hence we get the associated Shimura variety; for a sufficiently small open compact subgroup C of $G(\mathbb{A}_{\mathrm{f}})$ the corresponding Shimura variety $\mathcal{S}h_C$ is defined by

$$\mathcal{S}h_C = G(\mathbb{Q}) \setminus X_\infty \times G(\mathbb{A}_{\mathrm{f}})/C, \tag{2.5.2}$$

where X_∞ is the set of all conjugates of h under the action of $G(\mathbb{R})$. Note that X_∞ is isomorphic to the open unit-ball in $\mathbb{C}^2$. The Shimura variety $\mathcal{S}h_C$ is a finite disjoint union of quasi-projective manifolds, which are arithmetic quotients of the complex unit-ball. It is easy to see that the Shimura field E coincides with $\varepsilon(K)$.

We therefore obtain the canonical model Sh_C, which is a quasi-projective variety over $E = \mathbb{Q}(\sqrt{-7})$.

3. Mumford's fake projective plane

3.1. In this section we briefly review the Mumford's construction of a fake projective plane. We mainly focus on the construction of the discrete group Γ_{Mum} which gives the uniformization of the fake projective plane. For more details the reader should consult Mumford's original paper [Mu79].

We use the notation as in 2.1. Regarding L as a 3-dimensional K-vector space we define a hermitian form $h\colon L \times L \to \mathbb{Q}$ by $h(x,y) = \mathrm{tr}_{L/K}(x\overline{y})$ for $x, y \in L$. Subject to the basis $1, \zeta, \zeta^2$ of L the form h is represented by the matrix

$$H = \begin{bmatrix} 3 & \overline{\lambda} & \overline{\lambda} \\ \lambda & 3 & \overline{\lambda} \\ \lambda & \lambda & 3 \end{bmatrix}, \tag{3.1.1}$$

which is a positive definite hermitian matrix of determinant 7. Note that there exists a decomposition over $\mathbb{Q}$

$$H = WW^* \quad \text{with} \quad W = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ \mu & 0 & \lambda \end{bmatrix}. \tag{3.1.2}$$

Let $\mathrm{M}_3(K)$ be the set of all 3×3 matrices with entries in K, regarded as an algebra over $\mathbb{Q}$. We denote by $*\colon \mathrm{M}_3(K) \to \mathrm{M}_3(K)^{\mathrm{op}}$ the standard involution $(a_{ij})^* = (\overline{a}_{ji})$. We define another involution (positive of second kind) $\dagger\colon \mathrm{M}_3(K) \to \mathrm{M}_3(K)^{\mathrm{op}}$ by $A^\dagger = HA^*H^{-1}$.

3.2. Definition. We define an algebraic group I over $\mathbb{Q}$ by

$$I(R) = \{\gamma \in (\mathrm{M}_3(K) \otimes R)^\times \mid \gamma\gamma^\dagger \in R\}$$

for any (commutative) $\mathbb{Q}$-algebra R.

3.3. Lemma. *We have the following isomorphisms*:

1) $I(\mathbb{R}) \cong \mathrm{GU}(3)$. *In particular,* $I_{\mathrm{ad}}(\mathbb{R})$ *is a compact Lie group.*

2) $I(\mathbb{Q}_2) \cong \{(x,y) \in \mathrm{GL}_3(\mathbb{Q}_2) \times \mathrm{GL}_3(\mathbb{Q}_2)^{\mathrm{op}} \mid xy \in \mathbb{Q}_2\}$. *Hence in particular we have* $I_{\mathrm{ad}}(\mathbb{Q}_2) \cong \mathrm{PGL}_3(\mathbb{Q}_2)$.

Proof. 1) is clear by definition. To prove 2) we first note the isomorphism

$$\mathrm{M}_3(K) \otimes \mathbb{Q}_2 \cong \mathrm{M}_3(\mathbb{Q}_2) \times \mathrm{M}_3(\mathbb{Q}_2)^{\mathrm{op}}$$

such that the involution $*$ acts on the right-hand side by interchanging the factors, i.e. $(x,y)^* = (y,x)$. We have $(x,y)^\dagger = (HyH^{-1}, H^{-1}xH)$. Then 2) follows easily from the existence of the decomposition (3.1.2). $\square$

3.4. Let $\mathcal{O}_K$ (resp. $\mathcal{O}_L$) be the integer ring of K (resp. L). We consider $\mathcal{O}_L$ as a free $\mathcal{O}_K$-module of rank 3. For a finite set S of prime numbers we denote by $\mathbb{A}_{\mathrm{f}}^S$

the prime-to-S part of the finite adele ring of $\mathbb{Q}$, and set

$$\widehat{\mathbb{Z}}^S = \prod_{\ell \notin S} \mathbb{Z}_\ell.$$

For S as above and a prime p we define the maximal open compact subgroups

$$\begin{aligned} C^S_{\max} &= \{\gamma \in I(\mathbb{A}^S_{\mathrm{f}}) \,|\, \gamma(\mathcal{O}_L \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}^S) \subseteq \mathcal{O}_L \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}^S\}, \\ C^{\max}_p &= \{\gamma \in I(\mathbb{Q}_p) \,|\, \gamma(\mathcal{O}_L \otimes_{\mathbb{Z}} \mathbb{Z}_p) \subseteq \mathcal{O}_L \otimes_{\mathbb{Z}} \mathbb{Z}_p\} \end{aligned}$$

of $I(\mathbb{A}^S_{\mathrm{f}})$ and $I(\mathbb{Q}_p)$, respectively.

We are going to define an open compact subgroup C_7 of $C^{\max}_7$. Since the prime 7 ramifies in K we have

$$\mathcal{O}_L \otimes_{\mathbb{Z}} \mathbb{Z}_7 \cong \widetilde{\mathbb{Z}}_7 \cdot 1 \oplus \widetilde{\mathbb{Z}}_7 \cdot \zeta \oplus \widetilde{\mathbb{Z}}_7 \cdot \zeta^2, \tag{3.4.1}$$

where $\widetilde{\mathbb{Z}}_7$ is the ramified quadratic extension of $\mathbb{Z}_7$. We can therefore consider the modulo $\sqrt{-7}$ reduction of $C^{\max}_7$, which takes values in $\mathrm{GL}_3(\mathbb{F}_7)$. Now the matrix $(H \bmod \sqrt{-7})$ is of rank 1 and has 2 dimensional null space, to which the action of $C^{\max}_7$ can be restricted. We therefore obtain a homomorphism

$$\varpi \colon C^{\max}_7 \longrightarrow \mathrm{GL}_2(\mathbb{F}_7). \tag{3.4.2}$$

We take a 2-Sylow subgroup P of the subgroup $\mathrm{GL}_2(\mathbb{F}_7) \bigcap \{\gamma \,|\, \det \gamma = \pm 1\}$ and put $C_7 = \varpi^{-1}(P)$.

3.5. Definition. Set $C^2_{\mathrm{Mum}} = C^{2,7}_{\max} C_7$, which is an open compact subgroup of $I(\mathbb{A}^2_{\mathrm{f}})$. We define the subgroup Γ_{Mum} of $I_{\mathrm{ad}}(\mathbb{Q}_2) \cong \mathrm{PGL}_3(\mathbb{Q}_2)$ to be the image of

$$I(\mathbb{Q}) \bigcap (I(\mathbb{Q}_2) \times C^2_{\mathrm{Mum}}) \longrightarrow I_{\mathrm{ad}}(\mathbb{Q}_2)$$

induced by the first projection, where the intersection on the left-hand side is taken in $I(\mathbb{A}_{\mathrm{f}})$.

3.6. Theorem (Mumford [Mu79]). *The subgroup Γ_{Mum} is a uniform lattice in $\mathrm{PGL}_3(\mathbb{Q}_2)$ which acts transitively on the vertices of the Bruhat-Tits building attached to $\mathrm{PGL}_3(\mathbb{Q}_2)$. Moreover, the quotient*

$$\mathcal{X}_{\mathrm{Mum}} = \Gamma_{\mathrm{Mum}} \backslash \Omega^3_{\mathbb{Q}_2} \tag{3.6.1}$$

of the Drinfeld symmetric space $\Omega^3_{\mathbb{Q}_2}$ is algebraized to a fake projective plane X_{Mum}.

Proof. We need to show that the subgroup Γ_{Mum} coincides with the one constructed by Mumford in [Mu79]. The Mumford's group is the image of

$$I_1(\mathbb{Q}) \bigcap (I(\mathbb{Q}_2) \times C^2_{\mathrm{Mum}}) \longrightarrow I_{\mathrm{ad}}(\mathbb{Q}_2),$$

where $I_1(\mathbb{Q}) = \{\gamma \in I(\mathbb{Q}) \,|\, \gamma\gamma^\dagger = 1\}$. Hence it suffice to see that every element $\gamma \in I(\mathbb{Q}) \bigcap (I(\mathbb{Q}_2) \times C^2_{\mathrm{Mum}})$ can be written as $\gamma = k\theta$ with $k \in \mathbb{Q}^\times$ and $\theta \in I_1(\mathbb{Q}) \bigcap (I(\mathbb{Q}_2) \times C^2_{\mathrm{Mum}})$. Set

$$k = (\gamma\gamma^\dagger)^{-1} \det \gamma$$

and $\theta = k^{-1}\gamma$. Since (γ mod $\sqrt{-7}$) $\in P$ and ($\dagger$ mod $\sqrt{-7}$) is trivial we have (k mod $\sqrt{-7}$) $\in P$. Hence we have (θ mod $\sqrt{-7}$) $\in P$. The property $\theta\theta^\dagger = 1$ follows from the fact that for any $\gamma \in I(\mathbb{Q})$ we have $\det\gamma\overline{\det\gamma} = (\gamma\gamma^\dagger)^3$. $\square$

4. Relation between X_{Mum} and Sh

4.1. Proposition. *Let I and G be the $\mathbb{Q}$-algebraic groups defined in* 3.2 *and* 2.4, *respectively. Then we have the following*:

(a) *I is an inner form of G.*

(b) $I(\mathbb{A}_\mathrm{f}^2) \cong G(\mathbb{A}_\mathrm{f}^2)$.

Moreover, the algebraic group I is uniquely determined up to $\mathbb{Q}$-isomorphisms by (a), (b) *and the following condition*:

(c) $I(\mathbb{Q}_2) \cong \{(x, y) \in \mathrm{GL}_3(\mathbb{Q}_2) \times \mathrm{GL}_3(\mathbb{Q}_2)^{\mathrm{op}} \mid xy \in \mathbb{Q}_2\}$.

Proof. First we note that, since $G(\mathbb{Q}_2) \not\cong I(\mathbb{Q}_2)$, the condition (a), (b) and (c) imply that $I_{\mathrm{ad}}(\mathbb{R})$ is a compact Lie group. By this the uniqueness of I will follow if we prove that the group G satisfies the Hasse principle; for this we refer the argument by Kottwitz [Ko92, §7]. Our situation belongs to the case (A) in the notation as in [Ko92, §5]. Kottwitz showed that validity of the Hasse principle for G depends only on that for the center $Z(G)$ of G. In our case we have $Z(G) = \mathrm{Res}_{K/\mathbb{Q}}\mathbb{G}_\mathrm{m}$, which clearly satisfies the Hasse principle, since K is a quadratic extension of $\mathbb{Q}$.

Next we prove that I is an inner form of G. It suffices to show that the $\mathbb{Q}$-algebra with involution $(\mathrm{M}_3(K), \dagger)$ is an inner form of $(D, \bigstar)$. Set

$$R = \mathrm{M}_3(K) \otimes L,$$

and consider the action ρ of the Galois group $\mathrm{Gal}(L/\mathbb{Q})$ on R via the second factor. The invariant part R^ρ is isomorphic to $\mathrm{M}_3(K)$. Note that the Galois group $\mathrm{Gal}(L/\mathbb{Q})$ is generated by σ and complex conjugation (cf. 3.1). Now we consider the matrix

$$Q = \Phi(\Pi) = \begin{bmatrix} & & \mu \\ 1 & & \\ & 1 & \end{bmatrix} \in \mathrm{M}_3(K),$$

and define another Galois action ρ' by ρ'(complex conj.) $= \rho$(complex conj.) and $\rho'(\sigma)(A \otimes z) = (Q^{-1}AQ) \otimes z^\sigma$ for $A \in \mathrm{M}_3(K)$ and $z \in L$.

4.1.1. Claim. *The invariant part $R^{\rho'}$ is isomorphic to D.*

Looking at the isomorphism $R \cong \mathrm{M}_3(L) \times \mathrm{M}_3(L)^{\mathrm{op}}$ such that σ (resp. complex conjugation) acts on $\mathrm{M}_3(L) \times \mathrm{M}_3(L)^{\mathrm{op}}$ diagonally (resp. by interchanging the factors), we deduce that $R^{\rho'}$ is isomorphic to

$$D' = \{A \in \mathrm{M}_3(L) \mid Q^{-1}A^\sigma Q = A\}.$$

Obviously D' contains $\Phi(D)$. Comparing $\mathbb{Q}$-ranks we have $D' = \Phi(D)$, thereby the claim.

4.1.2. Lemma. *Let A be a ring. For $u \in A^\times$ with $u^*u^{-1} \in Z(A)$ we define the involution i_u on the ring $A \times A^{\mathrm{op}}$ by*

$$(x, y)^{i_u} = (uyu^{-1}, u^{-1}xu)$$

for $x, y \in A$. Then the ring with involution $(A \times A^{\mathrm{op}}, i_u)$ is isomorphic to $(A \times A^{\mathrm{op}}, i_1)$.

This is well-known. In fact, the conjugation by $(1, u)$ gives the desired isomorphism.

By the lemma we deduce in particular that the L-algebras with involution (R, i_H) and $(R, i_{\Phi(b)})$ are isomorphic to each other. Since $H \in \mathrm{M}_3(K)$, the involution i_H commutes with the Galois action ρ, and hence it induces an involution on $R^\rho = \mathrm{M}_3(K)$, which coincides with $\dagger$. Also, since $\Phi(b) \in \mathrm{M}_3(K)$ and $\Phi(b)Q = Q\Phi(b)$, the involution $i_{\Phi(b)}$ commutes with the Galois action ρ' and can be restricted to $R^{\rho'} = D$, which is nothing but $\bigstar$. We therefore deduce that $(\mathrm{M}_3(K), \dagger)$ is an inner form of $(D, \bigstar)$, thereby (a).

Finally we prove (b). Let $\ell \neq 2$ be a rational prime. By 2.1.3 we know that $D_\ell \cong \mathrm{M}_3(K) \otimes \mathbb{Q}_\ell$. Hence it suffices to show that the involutions $\dagger$ and $\bigstar$ are $\mathbb{Q}_\ell$-isomorphic to each other. If the prime ℓ splits in K, the assertion follows from 4.1.2. Suppose that ℓ does not split in K. In order to show that $\dagger$ and $\bigstar$ are isomorphic it suffices to show that the hermitian forms H and $H' = (\lambda - \overline{\lambda})\Phi(b)$ are isomorphic. We can do this by the well-known fact (cf. [MH73, App. 2]): An isomorphism classe of hermitian forms on a fixed vector space over a local field F (with involution $*$) is determined by the residue class of determinant modulo $\mathrm{N}_{F/F_0}(F^\times)$, where F_0 is the fixed field of $*$. We know that $\det H = 7$ and $\det H' = 7^2$. If $\ell \neq 7$ then $K_\ell/\mathbb{Q}_\ell$ is an unramified quadratic extension. Since both $\det H$ and $\det H'$ are unit integer and since every unit integer of $\mathbb{Q}_\ell$ is a norm of an element in the unramified quadratic extension, we have the desired result in this case. If $\ell = 7$ then $K_7 = \mathbb{Q}_7(\sqrt{-7})$, and hence both $\det H$ and $\det H'$ are obviously norms, thereby the result.

Now we have proved (b), and hence the proof of the proposition is finished. $\square$

4.2. We fix an isomorphism $\varphi\colon I(\mathbb{A}_{\mathrm{f}}^2) \cong G(\mathbb{A}_{\mathrm{f}}^2)$ as in 4.1 (b). Let $C_2^{\max} \subset G(\mathbb{Q}_2)$ be a maximal open compact subgroup defined by

$$C_2^{\max} = \{\gamma \in G(\mathbb{Q}_2) \,|\, (\mathcal{O}_D \otimes_{\mathbb{Z}} \mathbb{Z}_2)\gamma \subseteq \mathcal{O}_D \otimes_{\mathbb{Z}} \mathbb{Z}_2\}.$$

We define an open compact subgroup $C \subset G(\mathbb{A}_{\mathrm{f}})$ by

$$C = C_2^{\max}\varphi(C^2_{\mathrm{Mum}}) \tag{4.2.1}$$

(cf. 3.5). Let us fix an embedding $\nu\colon \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_2$. Let E_ν be the ν-adic completion of the Shimura field E $(= \varepsilon(K))$, and $\breve{E}_\nu$ the maximal unramified extension of E_ν. By [RZ96, 6.51] (see also [BZ95, 1.12][Va98a][Va98b]) and the uniqueness of I as in 4.1 we have an isomorphism of formal schemes

$$I(\mathbb{Q})\backslash(\widehat{\Omega}^2_{\mathbb{Q}_2} \times_{\mathrm{Spf}\,\mathbb{Z}_2} \mathrm{Spf}\,\mathcal{O}_{\breve{E}_\nu}) \times G(\mathbb{A}_{\mathrm{f}})/C \xrightarrow{\sim} \mathrm{Sh}^\wedge_C \times_{\mathrm{Spf}\,\mathcal{O}_{E_\nu}} \mathrm{Spf}\,\mathcal{O}_{\breve{E}_\nu}, \tag{4.2.2}$$

where $\mathrm{Sh}^\wedge_C$ is the formal completion of a model of Sh_C over $\mathrm{Spec}\,\mathcal{O}_{E_\nu}$ along the closed fiber.

4.3. Theorem. *The canonical model Sh_C of the Shimura variety $\mathcal{S}h_C$ has exactly three geometrically connected components defined over $\mathbb{Q}(\zeta_7)$ which are permuted*

by the action of $\mathrm{Gal}(\mathbb{Q}(\zeta_7)/\mathbb{Q}(\sqrt{-7}))$*. Moreover, for any connected component* S *of* Sh_C *there exists an isomorphism of schemes*

$$X_{\mathrm{Mum}} \times_{\mathrm{Spec}\,\mathbb{Q}_2} \mathrm{Spec}\,\mathbb{Q}_2(\zeta_7) \xrightarrow{\sim} S \times_{\mathrm{Spec}\,\mathbb{Q}(\zeta_7)} \mathrm{Spec}\,\mathbb{Q}_2(\zeta_7) \tag{4.3.1}$$

over $\mathbb{Q}_2(\zeta_7)$*, where* X_{Mum} *is the Mumford's fake projective plane* (*cf.* 3.6).

Proof. First we note that by [RZ96, 6.51] we can take the isomorphism as in (4.2.2) such that the natural descent datum on the right-hand side induces on the left-hand side the natural descent datum on the first factor multiplied with the action of

$$(\Pi, \Pi^{-1}) \in G(\mathbb{Q}_2) \subset D_\lambda^\times \times D_\lambda^{\times\mathrm{op}}$$

on $G(\mathbb{A}_\mathrm{f})/C$. Since $\Pi \in \mathcal{O}_{D_\lambda}$ and $\Pi^{-1} = \overline{\mu}\Pi^2 \in \mathcal{O}_{D_{\overline{\lambda}}}$ we have $(\Pi, \Pi^{-1}) \in C_2^{\max}$. Therefore the action of (Π, Π^{-1}) is trivial on $G(\mathbb{A}_\mathrm{f})/C$. Hence the isomorphism (4.2.2) descends to an isomorphism

$$I(\mathbb{Q})\backslash\widehat{\Omega}^2_{\mathbb{Q}_2} \times G(\mathbb{A}_\mathrm{f})/C \xrightarrow{\sim} \mathrm{Sh}^\wedge_C \tag{4.3.2}$$

over $\mathbb{Z}_2 \cong \mathcal{O}_{E_\nu}$.

Next we note that the left-hand side of (4.2.2) is the disjoint union of formal schemes of form

$$\Gamma_\gamma\backslash(\widehat{\Omega}^2_{\mathbb{Q}_2} \times_{\mathrm{Spf}\,\mathbb{Z}_2} \mathrm{Spf}\,\mathcal{O}_{\breve{E}_\nu}),$$

where $\Gamma_\gamma =$ the image of $I(\mathbb{Q})\bigcap(I(\mathbb{Q}_2) \times (\gamma C^2_{\mathrm{Mum}}\gamma^{-1}))$ in $I_{\mathrm{ad}}(\mathbb{Q})$ for $\gamma \in \mathbb{G}(\mathbb{A}^2_\mathrm{f})$. Note also that $\Gamma_1 = \Gamma_{\mathrm{Mum}}$ (cf. 3.5).

By these observations it suffices to prove that the canonical model Sh_C of the Shimura variety has three connected components as in the theorem. To do this we consider the torus T over $\mathbb{Q}$ defined by

$$T = \{(k, f) \in \mathrm{Res}_{K/\mathbb{Q}}\mathbb{G}_{\mathrm{m},K} \times \mathbb{G}_{\mathrm{m},\mathbb{Q}} \mid k\overline{k} = f^3\} \cong \mathrm{Res}_{K/\mathbb{Q}}\mathbb{G}_{\mathrm{m},K}$$

and define an epimorphism $\vartheta\colon G \to T$ by $\gamma \mapsto (\mathrm{N}^\circ_{D^{\mathrm{op}}/K}\,\gamma, c(\gamma))$, where $\mathrm{N}^\circ_{D^{\mathrm{op}}/K}$ is the reduced norm (the last $\cong$ is given by $(k, f) \mapsto kf^{-1}$). The kernel of ϑ is the derived group of G. By [De71, §2] we know that the morphism ϑ induces the canonical morphism of Shimura varieties

$$\mathcal{S}h_C = G(\mathbb{Q}) \setminus X_\infty \times G(\mathbb{A}_\mathrm{f})/C \longrightarrow T(\mathbb{Q})_+ \setminus T(\mathbb{A}_\mathrm{f})/\vartheta(C)$$

which induces the Galois-equivariant bijection between the sets of all connected components of canonical models of each side.

To calculate $\vartheta(C)$ we have only to look at the component $\varphi(C_7)$ at 7.

4.3.3. Claim. *Under the canonical identification* $T(\mathbb{Q}_7) \cong \widetilde{\mathbb{Q}}_7^\times$*, where* $\widetilde{\mathbb{Q}}_7^\times$ *is the ramified quadratic extension of* $\mathbb{Q}_7$*, we have*

$$\vartheta(\varphi(C_7)) \cong \pm 1 + \sqrt{-7}\,\widetilde{\mathbb{Z}}_7^\times,$$

where $\widetilde{\mathbb{Z}}_7^\times$ *is the integer ring of* $\widetilde{\mathbb{Q}}_7^\times$*.*

First we note that the map $\vartheta \circ \varphi$ is described by

$$\vartheta \circ \varphi : C_7 \ni \gamma \mapsto (\gamma\gamma^\dagger)^{-1} \det \gamma \in \widetilde{\mathbb{Z}}_7^\times.$$

Since $\pm 1 + 7\,\mathbb{Z}_7^\times \subset C_7$ we immediately deduce that $\vartheta(\varphi(C_7))$ contains $\pm 1 + 7\,\mathbb{Z}_7^\times$. Moreover, since $\pm 1 + \sqrt{-7}\,\widetilde{\mathbb{Z}}_7^\times \subset C_7$, $\vartheta(\varphi(C_7))$ contains the element $k^2\overline{k}^{-1}$ for any $k \in \pm 1 + \sqrt{-7}\,\widetilde{\mathbb{Z}}_7^\times \subset C_7$; but, since $k\overline{k} \in \vartheta(\varphi(C_7))$, we deduce that

$$(\pm 1 + \sqrt{-7}\,\widetilde{\mathbb{Z}}_7^\times)^3 = \pm 1 + \sqrt{-7}\,\widetilde{\mathbb{Z}}_7^\times \subseteq \vartheta(\varphi(C_7)).$$

Hence the claim follows if we check that $(\vartheta(\varphi(C_7)) \bmod \sqrt{-7}) = \{\pm 1\}$. By an argument similar to that in the proof of 3.6 any $\gamma \in C_7$ has a decomposition $\gamma = k\theta$ in C_7 with $\theta\theta^\dagger = 1$. Since $\det\theta \det\theta^\dagger \equiv (\det\theta)^2 \equiv 1 \bmod \sqrt{-7}$ we have $\vartheta(\varphi(\gamma)) \equiv \pm k$. But, since $k = \gamma\theta^{-1} \bmod \sqrt{-7}$ belongs to P, the order of k is either one or two; hence $k \equiv \pm 1 \bmod \sqrt{-7}$. We therefore get $\vartheta(\varphi(\gamma)) \equiv \pm 1 \bmod \sqrt{-7}$ for any $\gamma \in C_7$, which proves the claim.

Now in view of the theory of complex multiplication we find that the Shimura variety $T(\mathbb{Q})_+ \backslash T(\mathbb{A}_\mathrm{f})/\vartheta(C)$ has three connected components having field of definition $L = \mathbb{Q}(\zeta)$ which are permuted by $\mathrm{Gal}(L/K)$, since the ray class field corresponding to C is easily checked to be L. We therefore proved the all statements in the theorem. □

4.4. Corollary. *Each connected component of the complex surface $\mathcal{S}h_C$ is a fake projective plane, which has a field of definition $\mathbb{Q}(\zeta_7)$. Moreover, it is an arithmetic quotient of complex unit-ball.* □

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